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Modelos GARCH con tendencia y detección de puntos de cambio

Tesis presentada por

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Dedication

To my family

For being the main engine in this whole process and always supporting me from the distance.

To my professor

To Dr. Juan Manuel Burgos for recommending me a new thesis supervisor and Dr. Feliú Sagols for agreeing to take a class with him totally different than what I expected

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RESUMEN

Éste trabajo está centrado al estudio de un modelo autoregresivo que describe el comportamiento de la volatilidad de ciertas series de tiempo con una componente adicional: la tendencia. Los modelos GARCH o modelo de heterocedasticidad condicional autoregresiva generalizada, utilizan los términos σ^2 y ϵ^2 como estimadores. En los modelos GARCH con tendencia, añadimos la variable temporal y además, evaluaremos hipótesis con respecto al comportamiento de estos modelos.

Nos centraremos en hallar, lo que hemos llamado, *punto de cambio* y éste se calculará con ayuda de la prueba del cociente de máxima verosimilitud logarítmica. Éste punto de cambio será aquel que nos arroje un valor máximo en la valuación del cociente, basándonos en dos hipótesis; H_0: la serie se puede describir por medio de coeficientes fijos, contra H_1: existe un punto que separa la serie en dos y ajusta el modelo en un "antes" y un "después", que son descritos por coeficientes únicos en estos dos tiempos.

Tomaremos además, las series de tiempo asociadas a los precios del petróleo, WTI, las tasas de cambio de los principales países exportadores a nivel mundial y uno importador y, adicional, los índices de los mercados representativos del mundo, que para nuestro caso, será el índice S&P 500 de Estados Unidos, el índice de la bolsa mexicana de valores, MEXBOL o MXX y por último, el A300 de China.

ABSTRACT

This work is focused on the study of an autoregressive model that describes the behavior of the volatility of certain time series with an additional component: the trend. The GARCH models or generalized autoregressive conditional heteroscedasticity model, use the terms σ^2 and ϵ^2 as estimators. In the GARCH models with a trend, we add the temporal variable and also evaluate hypotheses regarding the behavior of these models.

We will focus on determining what we have called a *change point* and this will be calculated through the test of the maximum logarithmic likelihood ratio. This change point will be the point that gives us a maximum value in the coefficient valuation, based on two hypotheses; H_0: the series can be described with fixed coefficients, against H_1: there is a point(the change point) that split the series values and fit the model in two times; where each time is described with unique coefficients for each time.

We will also take the time series associated with oil prices, WTI, the exchange rates of the main exporting countries worldwide and one importing country, and in addition, the indices of the most representative markets in the world, which for our case, will be the S&P 500 index of the United States, the index of the Mexican stock exchange, MEXBOL or MXX and finally, the A300 index of China.

CHAPTER 1

PRELIMINARY

1.1 TIME SERIES

In this chapter, it will be shown some basic concepts about time series analysis and modelling.

When we talk about **time series** we refer to data that is recorded or collected at regular time intervals; could be daily, weekly or annual, it can vary depending on how often it is taken. They are used to study relationships between various variables that change over time and influence each other. From the probabilistic point of view a time series is a sequence of random variables indexed according to an increasing time parameter.

We can represent a time series as follows:

$$X = \{X_t : t \in T\}$$

where T is the index set.

1.1.1 TREND

The trend of a time series is given by the long-term general movement of the series. The long-term trend of many business series (industrial and commercial), such as sales, exports, and production, often approaches a straight line. This trend line shows that something is increasing or decreasing at a constant rate and we say that it has a **linear trend**. The method used to obtain the best-fit straight line is the Least Squares Method.

On the other hand, when the time series presents a curvilinear behavior, this behavior is said to be **nonlinear**. Among the nonlinear trends that can occur in a series are, the polynomial, logarithmic, exponential, potential or any other, and the methods to find the best fit for those behaviors can vary.

We next introduce some basic concepts and notations.

Definition 1.1 The *returns* of a time series are the differences $x_j - x_{j-1}$ for $j \in T$ and we represent this differences with ε_t.

Definition 1.2 The *volatility* (usually denoted by σ) of a time series is the degree of variation of a time series, usually measured by the standard deviation of logarithmic returns; i.e. the returns calculated over the logarithmic values of the series.

1.2 AR, ARMA and ARCH Models

There are some assumptions we can establish about particular time series and their behaviour. We're not going further on the statistic analysis at this part.

An $\mathbf{AR(p)}$ **model** (Autoregressive) is a representation of the behaviour of a time series. Here, the output variable depends linearly on its own previous values, at most, the previous p values. The model is defined as:

$$X_t = c + \sum_{i=1}^{p} \varphi_i X_{t-i} + \epsilon_t$$

where c is a constant value, $\varphi_1, \ldots, \varphi_p$ are the parameters determined by the model and ϵ_t is white noise.

This is a model that establishes a relationship between current data of the time series with the lagged time series.

An $\mathbf{ARMA(p, q)}$ **model** (Autoregressive Moving-Average) is a representation of the behaviour of a time series where the output variable depends linearly on, at most, the previous p values and the previous q values of white noise error terms. The model is defined as:

$$X_t = c + \epsilon_t + \sum_{i=1}^{p} \varphi_i X_{t-i} + \sum_{i=1}^{q} \theta_i \epsilon_{t-i}$$

where, c is a constant value, $\varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q$ are the parameters determined by the model and ϵ_t is white noise.

On the other hand, the $\mathbf{ARCH(q)}$ **model** (Autoregressive Conditional Heteroscedasticity) is a model that describes the behaviour of the variance of the error terms(return residuals) of a time series denoted by ϵ_t. These terms are split into two: w_t a stochastic piece and σ_t a time dependent standard deviation.

Here, we work with the previous q return terms. This is:

$$\varepsilon_t = \sigma_t w_t$$

And the model is not over the values of the series, but over the σ_t^2, following this structure:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

where $\alpha_0 > 0$ and $\alpha_i \geq 0, \forall i > 0$.

This type of model can be estimated using ordinary least squares.

1.3 GARCH MODEL

A **GARCH model** is an ARMA model for the error variance, where GARCH stands for "Generalized Autoregressive Conditional Heteroskedasticity". Here, the model can be described as follows:

$$\varepsilon_t = \sigma_t w_t$$

where $w_t \sim \mathcal{N}(0, 1)$, and:

$$\sigma_t^2 = c + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2$$

we refer to the model as a GARCH(p, q) model, where p is the order of the GARCH terms σ^2 and q is the order of the ARCH terms ε^2.

1.4 GARCH WITH TREND

So far, we have described the models that have been worked on since the 1920s, with the first formal appearance of AR models, and the GARCH model at 1986 with Bollerslev [4]. Now, with recent work by Guerrero et. al. [6], there was an approach to a new model based on GARCH model, where is considered an extra parameter, **the trend**.

At the Guerrero et. al. [6] work, they focus on an specific GARCH model: a GARCH(1,1) model where it is added a linear trend. The GARCH(1,1) model with trend can be described with the following dynamics:

$$\varepsilon_t = \sigma_t w_t \qquad w_t \overset{i.i.d}{\sim} \mathcal{N}(0,1)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta t + \gamma \sigma_{t-1}^2$$

where β is the coefficient that describes the relation with the trend, α_1 the coefficient that describes the relation with the error returns and γ the coefficient for the GARCH terms.

At Guerrero et. al. [6] work, one of the main reason to add the linear trend on their model was due to political and economical scenarios that directly influenced the behavior of the time series and let them consider the series in different time intervals.

1.5 MOTIVATION

Going into more details with the model explained in the previous section, the GARCH(1,1) with trend model, we want to analyze the time series where these changes in the model parameters can be identified, more precisely, in the series trend.

As we mentioned at the end of the previous section, Guerrero et. al. [6] focused their work on analyzing the behavior of the GARCH(1,1) model with trend in a set of values for the time series that were within similar political scenarios. On this work, we want to identify when these changes occur in the series, not based solely on political scenarios, but on changes inherent to the time series and try to find a relationship between the change point of the time series and different external events related to it.

To give an approach to the behavior of the model, and try to identify the changes, we will focus on the likelihood function, which, basically, measures the goodness of fit of a model to a sample of data for given values of the unknown parameters. We are going to face two likelihood function: the first, that measures the goodness of fit of the GARCH(1,1) with trend model (without break point), against the second, a likelihood function that assumes that the we have two GARCH(1,1) with trend models, assuming that we have n values: the first k values are described with parameters different than the last $n-k$ values, this is, we have a break point at k.

Then, we introduce a log-likelihood ratio function, where we face the two likelihood functions described in the previous paragraph, through a division, and we calculated the values of the parameters that maximize this ratio. Also, to simplify the calculations, we work with the logarithms of these functions.

One of the advantages offered by the assumption of the existence of a break point in the time series is, being able to identify possible relationships between external scenarios and their impact on the time series. Imagine, for example, the prices of a barrel of oil and the current pandemic caused by the coronavirus. Here, we could see that once the virus began to spread worldwide and people started to work from home, the price of oil began to fall.

We want to look further in the behaviour of the oil prices, and different exchange rates. The main reason for choosing exchange rates is the close relationship that exists between the economic base of certain countries and oil. Here, we will be able to analyze and determine that many countries, in which their economy is based on the commercialization of crude oil and oil, will undergo changes in the exchange rates between their local currencies with the dollar.

CHAPTER 2
MODEL

2.1 MODEL

At the following chapter, we will go into more details about the methodology to find the break point for the GARCH(1,1) with trend model. Here, we will work on the hypothesis over the returns:

$$\varepsilon_t = \mu + \sigma_t w_t \qquad\qquad w_t \overset{i.i.d}{\sim} \mathcal{N}(0,1)$$

and, we want to test the following hypotheses:

$$H_0 : \begin{cases} \mu = \mu_0 \\ \sigma_t^2 = \alpha_0 + \alpha_1(\varepsilon_{t-1} - \mu_0)^2 + \beta t + \gamma\sigma_{t-1}^2 \quad 1 \leq t \leq n \end{cases}$$

$$H_1 : \exists k, \text{s.t.} \begin{cases} \mu = \mu_1 \\ \sigma_t^2 = \alpha_0 + \alpha_1(\varepsilon_{t-1} - \mu_1)^2 + \beta_1 t + \gamma\sigma_{t-1}^2 \quad for\ t \leq k \\ \\ \mu = \mu_2 \\ \sigma_t^2 = \alpha_0 + \alpha_1(\varepsilon_{t-1} - \mu_2)^2 + \beta_2 t + \gamma\sigma_{t-1}^2 \quad for\ k+1 \leq t \leq n \end{cases}$$

As we can see, we have the null hypothesis, H_0, establishes that the volatility of the time series can be described with a GARCH(1,1) with trend model over all the set of values, with fixed parameters of α_0, α_1, β (the trend) and γ. On the other hand, with H_1 we establish that the model splits into "two periods", where they differ by the trend coefficient and the μ value for the returns hypotheses.

The previous statistical procedure could be used in the case in which a structural change in the parameters of the process X, with the following dynamics:

$$X_t = X_0 \exp\left\{\mu_i t + \sum_{j=1}^{t} \sigma_j w_j\right\} \qquad i = 1, 2$$

is suspected at some time $k \in [0, t]$. This is true since:

$$\log(X_t / X_{t-1}) = \mu_i + \sigma_t w_t$$

2.2 Log-Likelihood Radio Test

With the previous hypotheses H_0 and H_1, let's consider the Likelihood under those. First, under H_0 we have:

$$\begin{aligned} L_0(\mu_0, \alpha_0, \alpha_1, \beta, \gamma) &= \prod_{j=1}^{n} f(\varepsilon_j) \\ &= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_j^2(\beta)}} \exp\left\{-\frac{(\varepsilon_j - \mu_0)^2}{2\sigma_j^2(\beta)}\right\} \end{aligned}$$

alternatively, under H_1 we have:

$$\begin{aligned} L_1(\mu_1, \mu_2, \alpha_0, \alpha_1, \beta_1, \beta_2, \gamma, k) &= \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi\sigma_j^2(\beta_1)}} \exp\left\{-\frac{(\varepsilon_j - \mu_1)^2}{2\sigma_j^2(\beta_1)}\right\} \\ &\times \prod_{j=k+1}^{n} \frac{1}{\sqrt{2\pi\sigma_j^2(\beta_2)}} \exp\left\{-\frac{(\varepsilon_j - \mu_2)^2}{2\sigma_j^2(\beta_2)}\right\} \end{aligned}$$

It follows that the Log-likelihood ratio function is given by:

$$\lambda_k = \frac{L_0(\hat{\mu}_0, \hat{\alpha}_{00}, \hat{\alpha}_{01}, \hat{\beta}, \hat{\gamma}_0)}{L_1(\hat{\mu}_1, \hat{\mu}_2, \hat{\alpha}_{10}, \hat{\alpha}_{11}, \hat{\beta}_1, \hat{\beta}_2, \hat{\gamma}_1, k)}$$

For $k \in \{2, ..., n\}$.

In particular, we will use the following statistic to detect a change of drift:

$$\Lambda = \max_{2 \le k \le n} |-2\log(\lambda_k)|$$

To go further on details about Λ and λ_k's, let's take a look to the following chapter where we are going to find the λ_k's using Newton's method and explain the behaviour of the Λ approximately as a non-central ξ_n^2.

CHAPTER 3

METHODOLOGY

3.1 NEWTON'S METHOD

We will again consider the definition of the λ_k's and Λ introduced in Section 2.2. In order to evaluate the statistic Λ:

1. For $\hat{\mu}_0$, $\hat{\mu}_1$ and $\hat{\mu}_2$ from the λ_k, we have that:

$$\hat{\mu}_0 = \left(\sum_{j=1}^{n} \frac{\varepsilon_j}{\sigma_j^2}\right) \bigg/ \left(\sum_{j=1}^{n} \frac{1}{\sigma_j^2}\right), \; \hat{\mu}_1 = \left(\sum_{j=1}^{k} \frac{\varepsilon_j}{\sigma_j^2}\right) \bigg/ \left(\sum_{j=1}^{k} \frac{1}{\sigma_j^2}\right),$$

$$\hat{\mu}_2 = \left(\sum_{j=k+1}^{n} \frac{\varepsilon_j}{\sigma_j^2}\right) \bigg/ \left(\sum_{j=k+1}^{n} \frac{1}{\sigma_j^2}\right)$$

2. We seek a solution of the following:

$$\hat{\alpha}_{j0} \text{ solves } \frac{\partial \log(L_j)}{\partial \alpha_{j0}} = 0, \; j = 0, 1$$
$$\hat{\alpha}_{j1} \text{ solves } \frac{\partial \log(L_j)}{\partial \alpha_{j1}} = 0, \; j = 0, 1$$
$$\hat{\gamma}_j \text{ solves } \frac{\partial \log(L_j)}{\partial \gamma_j} = 0, \; j = 0, 1$$

and

$$\hat{\beta}_j \text{ solves } \frac{\partial (L_1)}{\partial \beta_j} = 0, \; j = 1, 2$$

Let $\Lambda_k = -2 \log \lambda_k$ and consider the hypotheses H_0 and H_1 from the previous chapter. As the likelihood functions are products of density functions for a normal distribution, the products will vary from 0 to 1, so on, the likelihood function.

Then, if the null hypothesis, H_0, is false (which means there is a change-point that occurs at some time k), implies that λ_k takes a small value (close to zero), which turns Λ_k to be a large value. As a result, we reject the null hypothesis in the case that the $|\Lambda_k|$ is too large, and the k, where the largest value, Λ_k, is observed, should be the time point where the change occurs.

We will estimate the parameters using standard Newton's method and explain why we choose the statistic Λ. In particular, we will focus on the parameter obtained from L_1 (for the estimation of the parameters under L_0, see Section 3 in Guerrero et. al.(2016a) [6]). By assumption, for the log likelihood we have:

$$\begin{aligned}
\log(L_1) = & \sum_{j=1}^{k} -\frac{1}{2}\left[\log(\sigma_j^2(\beta_1)) + \frac{(\varepsilon_j - \mu_1)^2}{\sigma_j^2(\beta_1)}\right] \\
& + \sum_{j=k+1}^{n} -\frac{1}{2}\left[\log(\sigma_j^2(\beta_2)) + \frac{(\varepsilon_j - \mu_2)^2}{\sigma_j^2(\beta_2)}\right]
\end{aligned}$$

In turn, if $\omega = (\alpha_0, \alpha_1, \sigma, \beta_1, \beta_2)^\top$, and using the definition itself of σ_t^2 and the gradient operator with the chain rule, we have then:

$$\begin{aligned}
\frac{\partial \log(L_1)}{\partial \omega} = & \sum_{j=1}^{k} -\frac{1}{2}\left[\frac{\frac{\partial \sigma_j^2(\beta_1)}{\partial \omega}}{\sigma_j^2(\beta_1)} - \frac{(\varepsilon_j - \mu_1)^2}{(\sigma_j^2(\beta_1))^2}\frac{\partial \sigma_j^2(\beta_1)}{\partial \omega}\right] \\
& + \sum_{j=k+1}^{n} -\frac{1}{2}\left[\frac{\frac{\partial \sigma_j^2(\beta_2)}{\partial \omega}}{\sigma_j^2(\beta_2)} - \frac{(\varepsilon_j - \mu_2)^2}{(\sigma_j^2(\beta_2))^2}\frac{\partial \sigma_j^2(\beta_2)}{\partial \omega}\right] \\
& = \sum_{j=1}^{k} \frac{1}{2\sigma_j^2(\beta_1)} k_j(\beta_1)\left(\frac{(\varepsilon_j - \mu_1)^2}{\sigma_j^2(\beta_1)} - 1\right) \\
& + \sum_{j=k+1}^{n} \frac{1}{2\sigma_j^2(\beta_2)} k_j(\beta_2)\left(\frac{(\varepsilon_j - \mu_2)^2}{\sigma_j^2(\beta_2)} - 1\right) \\
& =: \nabla L(\omega),
\end{aligned}$$

where

$$\begin{aligned}
k_j(\beta_1) &= (1, (\varepsilon_j - \mu_1)^2, \sigma_{j-1}^2, j, 0)^\top \\
k_j(\beta_2) &= (1, (\varepsilon_j - \mu_2)^2, \sigma_{j-1}^2, 0, j)^\top
\end{aligned}$$

Moreover, for the second moments we have that

$$\begin{aligned}\frac{\partial^2 L_1}{\partial\omega\partial\omega^\top} = &-\sum_{j=1}^{k}\frac{1}{2\sigma_j^4(\beta_1)}k_j(\beta_1)k_j^\top(\beta_1)\left(2\frac{(\varepsilon_j-\mu_1)^2}{\sigma_j^2(\beta_1)}-1\right)\\ &-\sum_{j=k+1}^{n}\frac{1}{2\sigma_j^4(\beta_2)}k_j(\beta_2)k_j^\top(\beta_2)\left(2\frac{(\varepsilon_j-\mu_2)^2}{\sigma_j^2(\beta_2)}-1\right)\end{aligned}$$

From the expression above, we define

$$\begin{aligned}J(\omega) &= \mathbb{E}\left[\frac{\partial^2 L_1}{\partial\omega\partial\omega^\top}\middle|\mathcal{F}_{j-1}\right].\\ &= -\sum_{j=1}^{k}\frac{1}{2\sigma_j^4(\beta_1)}k_j(\beta_1)k_j^\top(\beta_1)\\ &\quad -\sum_{j=k+1}^{n}\frac{1}{2\sigma_j^4(\beta_2)}k_j(\beta_2)k_j^\top(\beta_2)\end{aligned}$$

In summary, we estimate the parameters and iterations, for the Newton's method as follows:

$$\omega_{k+1} = \omega_k - J^{-1}(\omega_k)\nabla L(\omega_k).$$

3.2 THE LOG-LIKEHOOD STATISTIC

In this section we analyze the likelihood ratio statistic described in the previous chapter. We will show that this statistic behaves approximately as a non-central ξ_n^2. To this end, remember we set Λ_k as:

$$\Lambda_k = -2\log(\lambda_k)$$

We first note that, under H_0, the following should hold:

$$\begin{aligned}\sigma_j^2 &= \alpha_0 + \alpha_1\sigma_{j-1}^2\omega_{j-1}^2 + \beta t + \gamma\sigma_{j-1}^2\\ \sigma_j^2(\beta_1) &= \alpha_0 + \alpha_1\sigma_{j-1}^2\omega_{j-1}^2 + \beta_1 t + \gamma\sigma_{j-1}^2(\beta_1)\end{aligned}$$

$$\sigma_j^2(\beta_2) = \alpha_0 + \alpha_1 \sigma_{j-1}^2 \omega_{j-1}^2 + \beta_2 t + \gamma \sigma_{j-1}^2(\beta_1)$$

Now, let's work with the statistic to see the behaviour under the H_0:

$$\begin{aligned}\Lambda_k = &\sum_{j=1}^{n}\left[\log(\sigma_j^2) + \omega_j^2\right] - \sum_{j=1}^{k}\left[\log(\sigma_j^2(\beta_1)) + \frac{\sigma_j^2\omega_j^2}{\sigma_j^2(\beta_1)}\right] \\ &- \sum_{j=k+1}^{n}\left[\log(\sigma_j^2(\beta_2)) + \frac{\sigma_j^2\omega_j^2}{\sigma_j^2(\beta_2)}\right]\end{aligned}$$

In turn, rearranging terms, we have that:

$$\begin{aligned}\Lambda_k = &\sum_{j=1}^{k}\left[\log\left(\frac{\sigma_j^2}{\sigma_j^2(\beta_1)}\right) + \left(\omega_j^2 - \frac{\sigma_j^2\omega_j^2}{\sigma_j^2(\beta_1)}\right)\right] \\ &+ \sum_{j=k+1}^{n}\left[\log\left(\frac{\sigma_j^2}{\sigma_j^2(\beta_2)}\right) + \left(\omega_j^2 - \frac{\sigma_j^2\omega_j^2}{\sigma_j^2(\beta_2)}\right)\right]\end{aligned}$$

Then, factoring:

$$\begin{aligned}\Lambda_k = &\sum_{j=1}^{k}\left[\log\left(\frac{\sigma_j^2}{\sigma_j^2(\beta_1)}\right) + \omega_j^2\left(1 - \frac{\sigma_j^2}{\sigma_j^2(\beta_1)}\right)\right] \\ &+ \sum_{j=k+1}^{n}\left[\log\left(\frac{\sigma_j^2}{\sigma_j^2(\beta_2)}\right) + \omega_j^2\left(1 - \frac{\sigma_j^2}{\sigma_j^2(\beta_2)}\right)\right]\end{aligned}$$

Furthermore, given a first order Taylor approximation of the log function, it yields

$$\begin{aligned}\Lambda_k = &\sum_{j=1}^{k}\left[-1\left(1 - \frac{\sigma_j^2}{\sigma_j^2(\beta_1)}\right) + \omega_j^2\left(1 - \frac{\sigma_j^2}{\sigma_j^2(\beta_1)}\right)\right] \\ &+ \sum_{j=k+1}^{n}\left[-1\left(1 - \frac{\sigma_j^2}{\sigma_j^2(\beta_2)}\right) + \omega_j^2\left(1 - \frac{\sigma_j^2}{\sigma_j^2(\beta_2)}\right)\right]\end{aligned}$$

Then

$$\Lambda_k = \sum_{j=1}^{n} \left(1 - \frac{\sigma_j^2}{\hat{\sigma_j^2}} \right) (\omega_j^2 - 1)$$

where

$$\hat{\sigma_j^2} = \begin{cases} \sigma_j^2(\beta_1) & for\ j \leq k \\ \sigma_j^2(\beta_2) & for\ k+1 \leq j \leq n \end{cases}$$

Hence, as $\Lambda = \max(\Lambda_k)$, it follows, that in law:

$$\Lambda \approx \chi_n^2 - n$$

That is, Λ behaves approximately as a non-central χ_n^2. Thus:

$$\mathbb{E}\left[\chi_n^2 - n\right] = 0 \text{ and } Var\left[\chi_n^2 - n\right] = 2 \cdot n.$$

Chapter 4
Data

4.1 The Oil Prices and Exchange Rates Relationship

Based on previous research, it has been possible to establish a close relationship between real oil prices and real exchange rates("real" values are computed by dividing the the nominal value by the ratio of the Consumer Price Index (CPI)) in countries that have oil as an export product, such as Brazil, Canada, Mexico, Norway and Russia; in addition to a country that imports it, which is Peru. Even more, Amano and Norden (1998)[3] showed that oil price shocks impact US real effective exchange rates, and not the converse. This can be translated in the fact that oil price fluctuations play a major role explaining real exchange rate behaviour. In fact, they found that oil prices have a high forecasting power on US real effective exchange rate. Rautava (2004)[10] studied the same relationship for an oil-producer economy, Russia. He shows how oil price fluctuations influence the Russian ruble through the long-run equilibrium conditions and the short-run direct impacts.

Akram (2004)[1] found a strong nonlinear negative relationship between the value of the Norwegian krone and crude oil prices. He demonstrates that the incorporation of non-linearities can substantially improve the forecasting performance of structural exchange rate models. More specifically, variations in oil prices have important effects on exchange rates whenever oil prices are particularly low. In fact when oil prices are falling the effect is even higher.

Reboredo (2012)[11] analyzes the dependence structure of the co-movement between crude oil prices and exchange rates from 2000 to 2010 for seven countries. He uses correlations and copulas with the idea of capturing better the tail dependence. So, increments in oil prices come together with depreciation and vice versa. The intensity of this co-movement differs across currencies, oil-exporting countries like Canada, Norway and Mexico record more intense co-movement, while oil-importing countries show between null or weak relationship. In particular, the author shows the existence of tail independence between oil prices and exchange rates in the periods before and after the financial crisis.

This is going to be really important; we can take this as a step towards studying the behavior of its most recent price series. Recall that in the course of 2020, oil prices have had a rather unusual behavior given the current context of a global pandemic (Coronavirus disease, COVID-19), which has led many analysts and investors to rethink strategies to take advantage of these assets. Without going too far, Ferraro et al. (2015)[5] show the short-run relationship between variations of oil prices and variations of the nominal exchange rate for countries like Canada and Norway. For these countries, Ferraro et al. (2015)[5] use daily observations and show that oil prices do forecast nominal exchange rates. In particular, the daily oil price model outperforms the monthly and quarterly oil price models. They conclude that the better predictive power of the model is due to the frequency instead of the length of the series. Finally, Jiranyakul (2015)[7] shows that oil price volatility causes real exchange rate volatility in Thailand.

Our research uses the widely documented relationship oil prices and exchange rates and extend some findings on volatility. Our research is related to Ferraro et al. (2015)[5] in the sense that we analyze the nominal exchange rate and use real-time data. However, we focus on the volatility of the series and in particular we detect changes of drifts in the volatility. We answer the question whether a change of drift in volatility of the oil prices can precede a change of drift in volatility of exchange rates for 6 oil-exporting countries and 1 oil-importing country just to contrast the results. In turn, this results may inform policy and eventually support the the design of preventive strategies.

4.2 DATA AND DESCRIPTION

Our research includes empirical tests for the change of drift in the volatility of oil prices and exchange rates for 5 oil-exporting countries (Brazil, Canada, Mexico, Norway and Russia) and 1 oil-importing country, Peru. For the analysis, we use daily observations ranging from October 1st, 2019 to October 1st, 2020 of the spot price for the Brent and WTI crude oils. We recall that the WTI is usually used as the benchmark in USA and as the underlying commodity of the NYSE for oil futures contracts. All series with exception of WTI were obtained from Alpha Vantage databases [2]. WTI info came from Quandl database [9].

In addition, we will see if there was any impact that these changes had on the volatility of the WTI with the indexes that represent the great markets of the world, such as the S&P 500 (USA), S&P / BMV IPC (Mexico) and S&P China A 300 by the same period. All of their information, came from Yahoo Finance [12].

Since oil and exchange rates co-move in small open economies, as long as oil price shocks are within a given range, central banks concerned with price stability may resist exchange rate fluctuations by adjusting interest rates. However, shocks outside of the range may lead to high or low interest rates, which could destabilise the economy by reducing external competitiveness and weaken their financial system. In such cases, gains from allowing the exchange rate to fluctuate may exceed those of keeping it stable. Thus, when low oil prices are a source of economic fluctuations, monetary authorities may abandon their commitment to avoid exchange rate depreciation allowing the exchange rate to fluctuate.

Most of the studied countries, with exception of Peru, are oil-exporting and small open economies whose relative small share in the global oil market allows the assumption of price-taker countries. For instance, Crude oil was the world's top export product in 2018, with Russia, Canada and USA as the top 10 exporters. Further statistics regarding these countries can be found at Chapter 6.

Figure 4.1 suggests the existence of a negative correlation between WTI oil prices and exchange rates for all countries. However, the intensity of the relationship appears to differ across countries. For the advanced oil-exporting countries Norway and Canada, analyzed in this work, the relationship appears to be stronger for extreme values and weaker in intermediate values. In turn, for emerging and oil-exporting countries like Mexico, Brazil and Russia the relationship appears to be strong even for intermediate values of oil price and exchange rates, especially for Mexico. In contrast, for the Peruvian sol the relationship appears to be weaker. Also, we can see an outlier of a negative price of WTI, as it seems to be the only value that don´t have any relation with the behaviour of the WTI prices.

However, we can see the relationships figure 4.2, between the oil prices and top Market indexes that represents the ”real status” of the market regarding their country; we can see that the S&P / BMV IPC of Mexico has a strong relationship with the crude oil prices, also positive and in the other side, the S&P 500 of USA and A300, the equivalent of the index for China, appears to have weaker relationships but the S&P 500 index seems to have a positive correlation.

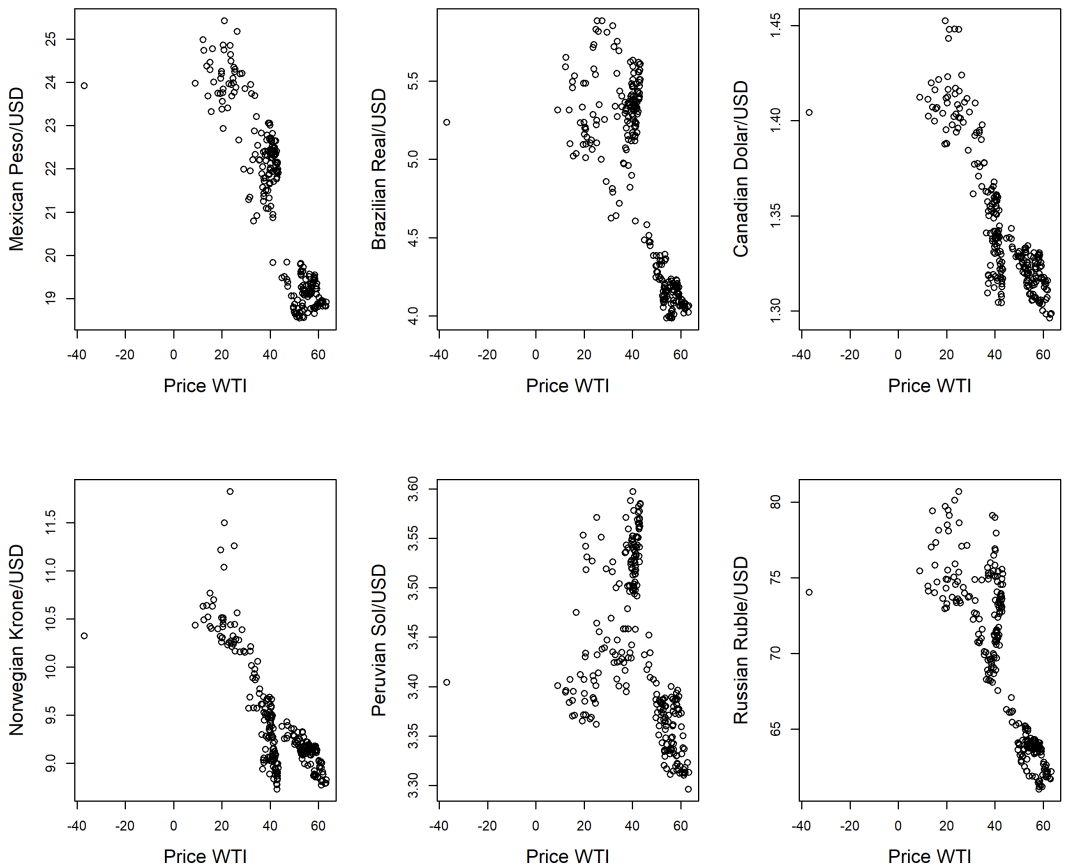

Figure 4.1: Scatterplot WTI vs. Currencies

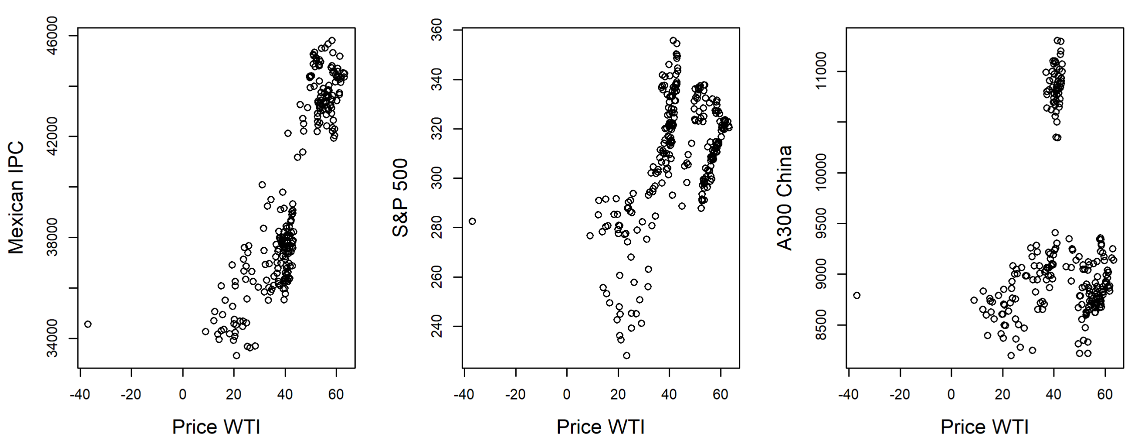

Figure 4.2: WTI vs. Market Indexes

CHAPTER 5

TEST RESULT AND ANALYSIS

5.1 MODEL TESTING

Since we were able to see a relationship between exchange rates and oil prices, we are going to evaluate each time series and determine the influence between the change in drift in oil volatility and the change in drift in volatility in oil prices. series. Recall that we also add an analysis on the most important market indices.

In table 6.1 we present the dates on which the greatest change detected occurred in the period from 2019/10 to 2020/09. Then, in columns 3 to 5, we have the p-values of the Breusch Pagan homogeneity test for the different trend-GARCH, GARCH (1,1) and ARCH (1) models. In columns 6 and 7, we have the estimate of the trend in the model before and after the change in the drift in volatility. Columns 8 and 9, characterize the trend of the process before and after the change in the drift in volatility.

Table 5.1: Statistical results

		Breusch-Pagan (P-Value)			Volatility drift (coef.)		Drift (coef.)	
	Change	T-GARCH	GARCH	ARCH	Drift Vol. B.	Drift Vol. A.	Drift B.	Drift A.
WTI	24/06/2020	0.90	0.11	0.18	4.09e-06	5.99e-07	0.10	0.14
Mexico	17/02/2020	0.34	0.43	< 0.001	-4.16e-08	8.20e-08	-0.12	0.10
Brazil	21/02/2020	0.54	0.01	< 0.001	-2.56e-07	5.94e-07	-0.03	0.08
Canada	25/02/2020	0.09	0.11	0.007	-1.03e-07	1.49e-07	0.10	0.11
Norway	16/01/2020	0.44	0.70	< 0.001	-5.55e-08	4.75e-08	-0.02	0.08
Peru	22/02/2020	0.74	0.22	0.12	-1.94e-08	-1.72e-09	-0.55	0.46
Russia	25/12/2019	0.83	0.04	0.04	-1.22e-08	4.79e-08	-0.18	0.06
SPY	19/02/2020	0.16	0.08	0.02	8.02e-08	2.88e-07	0.11	-0.09
MXX	08/06/2020	0.35	0.44	0.25	1.7e-06	2.61e-07	0.05	0.07
A300	29/06/2020	0.02	0.01	0.03	5.77e-07	1.09e-06	-0.04	0.27

5.2 INSIGHTS

According to this, p-values for the ARCH model show that the homoscedasticity can be rejected for almost all the countries and indexes, we can't reject the hypothesis for WTI, Peru and Market Index of Mexico. On the other side, for the GARCH model, it fails in three of the ten series (2 exchange rate series Brazil, Russia and the A300 of China).

In contrast, the Breusch-Pagan p-values for the trend-GARCH, show that the null hypothesis of homoscedasticity can not be rejected, except with A300 of China. Therefore, the trend-GARCH model mostly does outperform the ARCH and GARCH model, since it is able to detect the change of drift and is more accurate in capturing existing changes in the pattern of volatility.

We first analyze the case of the WTI crude oil prices in Figure 1 at Appendix A. For the WTI prices, the standardized residuals of the ARCH(1), GARCH(1,1) and Trend-GARCH models are homoscedastic. And, for the WTI oil price, our test detects a major peak on 24/06/2020.

Trend is positive and significant before and after the first change point (24/06/2020), it only slightly decrease. The standardized residuals with the trend-GARCH with change point are homoscedastic with p-value=0.903.

For the Mexican Peso/USD, the standardized residuals resulting of the Trend-GARCH with change point and GARCH(1,1) are homoscedastic, while for the standardized residuals of the ARCH(1) fail the Breusch-Pagan test. Changes on the drift of volatility are detected on 17/02/2020, where the drift goes from negative and significant to positive and significant. Also, some of the local peaks that have WTI, matches with the exchange rate, for example, at 15/01/2020 and 17/02/2020.

For the Brazilian Real/USD, the standardized residuals resulting of the Trend-GARCH with change point are homoscedastic, while for the standardized residuals of the ARCH(1) and GARCH(1,1) fail the Breusch-Pagan test. Changes on the drift of volatility are detected on 21/02/2020, where the drift goes from negative and significant to positive and significant. Here, the peak occurs 3 days after a local peak of WTI at 18/02/2020.

For the Canadian Dollar/USD, the standardized residuals resulting of the Trend-GARCH with change point and GARCH(1,1) are homoscedastic, while for the standardized residuals of the ARCH(1) fail the Breusch-Pagan test. Changes on the drift of volatility are detected on 25/02/2020, where the drift goes from negative and significant to positive and significant. Here, the peak occurs at almost a week after a local peak of WTI at 18/02/2020.

For the Norwegian Krone/USD, the standardized residuals resulting of the Trend-GARCH with change point and GARCH(1,1) are homoscedastic, while for the standardized residuals of the ARCH(1) fail the Breusch-Pagan test. Changes on the drift of volatility are detected on 16/02/2020, where the drift goes from negative and significant to positive and significant.

For the Peruvian Sol/USD, the standardized residuals resulting of the Trend-GARCH with change point, ARCH(1) and GARCH(1,1) are homoscedastic. Changes on the drift of volatility are detected on 22/02/2020, where the drift goes from negative and significant to positive and significant. Here, the peak occurs 3 days after a local peak of WTI at 18/02/2020.

For the Russian Ruble/USD, the standardized residuals resulting of the Trend-GARCH with change point are homoscedastic, while for the standardized residuals of the ARCH(1) and GARCH(1,1) fail the Breusch-Pagan test. Changes on the drift of volatility are detected on 25/12/2019, where the drift goes from negative and significant to positive and significant. Here, the peak matches with a local peak that occur with Mexican Peso/USD at 25/12/2019.

On the other side, going with the indexes market, we can see that, for the SPY index, the standardized residuals resulting of the Trend-GARCH and GARCH(1,1) with change point are homoscedastic, while for the standardized residuals of the ARCH(1) fail the Breusch-Pagan test. Trend is positive and significant before and after the first change point (19/02/2020), it only increase.

For the MXX index, the standardized residuals resulting of the Trend-GARCH with change point, ARCH(1) and GARCH(1,1) are homoscedastic. Trend is positive and significant before and after the first change point (08/06/2020), it only decrease. Here, the peak occurs at just three days away a local peak of SPY that occurs at 11/06/2020.

For the A300 index, none of the test tell us that the homoscedasticity hypothesis can't be rejected. Here, the peak occurs at just two weeks after a local peak of SPY that occurs at 11/06/2020.

Also, we can observe that the trend-GARCH fits better the data in almost all the cases since it adequately captures changes in trend volatility in comparison to the ARCH(1) and the GARCH (1,1) models.

Chapter 6
Conclusions and Expectations

6.1 Conclusions

After having evaluated our daily frequency time series, and having analyzed the (negative) relationship between oil prices and different exchange rates, our empirical results may suggest that a drift change in the volatility of oil prices precedes the drift change in the volatility of exchange rates given sudden changes in small intervals. To achieve our goal, we developed a statistical log probability ratio that detects drift changes in the Trend-GARCH model. We statistically quantify the effect of changes in the volatility of oil with changes in the volatility of currencies and we obtain a close relationship between the dates detected for exchange rates, with the dates detected for oil prices.

We also see that, although the maximum breaking point of oil is after the dates detected in the exchange rates, for the most part, their points of changes are very close to each other.

In addition to this, from the market indicators, which in fact have a (positive) relationship with the oil price, we can see that the changes in the volatility of the market indicators are affected in a similar way, since they share dates very close in which breaking points are registered (local and global). In this case, the relation of the market indicators don't seem to have hints of relation between the changes of drift in the volatility, expect with the SPY index.

6.2 Expectations

Given that our code demands to be able to calculate inverse of matrices for Newton's method iterations, this step threw some errors at the moment of the matrix inversion, i.e. singular matrices, which we believe is due to declaring initial values for the model parameters very far from the real / ideal of each break point.

We hope to obtain better results in the future if we implement a better selection of the initial values and minimize the possibility of errors in the calculation of the inverse matrix. Following the work done by Uribe et al.(2017) [8], were they proposed an ”Hybrid Metaheuristic” process to get better initial values in the parameters of the GARCH with trend model and be used in future works.

Since there are numerous variations of the GARCH models, which add other types of variables such as seasonality or rethink the model with other objects, for example, correlation matrices, we can find models that can be very useful. These models could even better describe the behavior of volatility in our time series and give us better statistics to find the break points.

This is mentioned, since the GARCH (1,1) model with a tendency proposed by Guerrero et. al [6] has not been worked on in other investigations, since these studies only mention the tendency to compare behaviors of the series in an informal way, but they do not use it as a variable within the model.

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Appendix A
Rates and WTI plots

It's important to see that we have some discontinuity over the graphs, the main reason for this, is due to the calculation of the inverse of a matrix related to specific prices/values over the dates that get us a singular matrix. On those cases, we simply omit the value for the test.

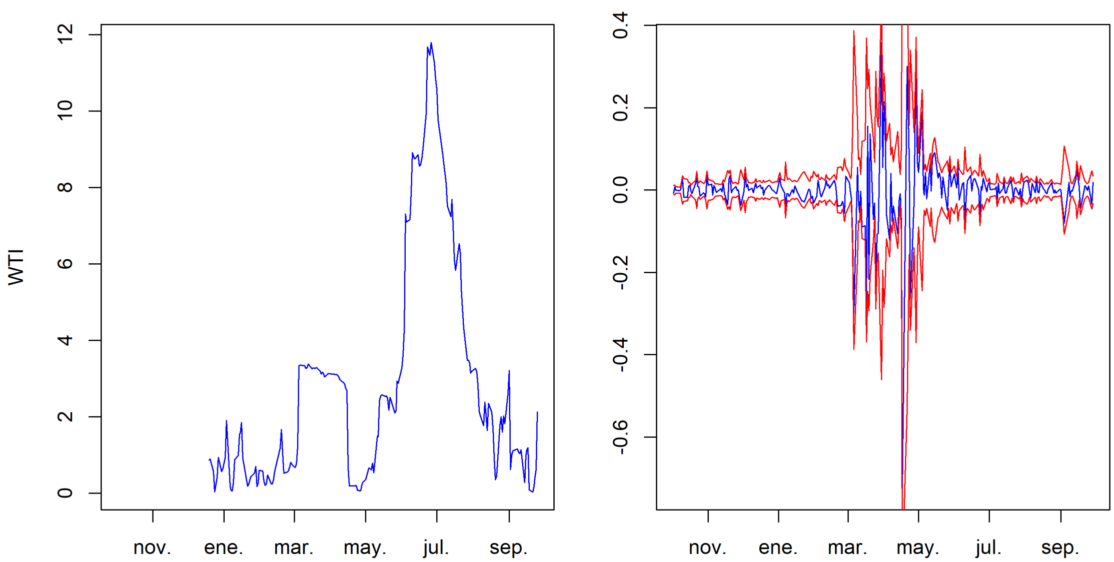

Fig. 1: Left: Test on WTI to detect the change point.
Right: Roots of the square of the volatility modelled by trend-GARCH model in red, volatility in blue.

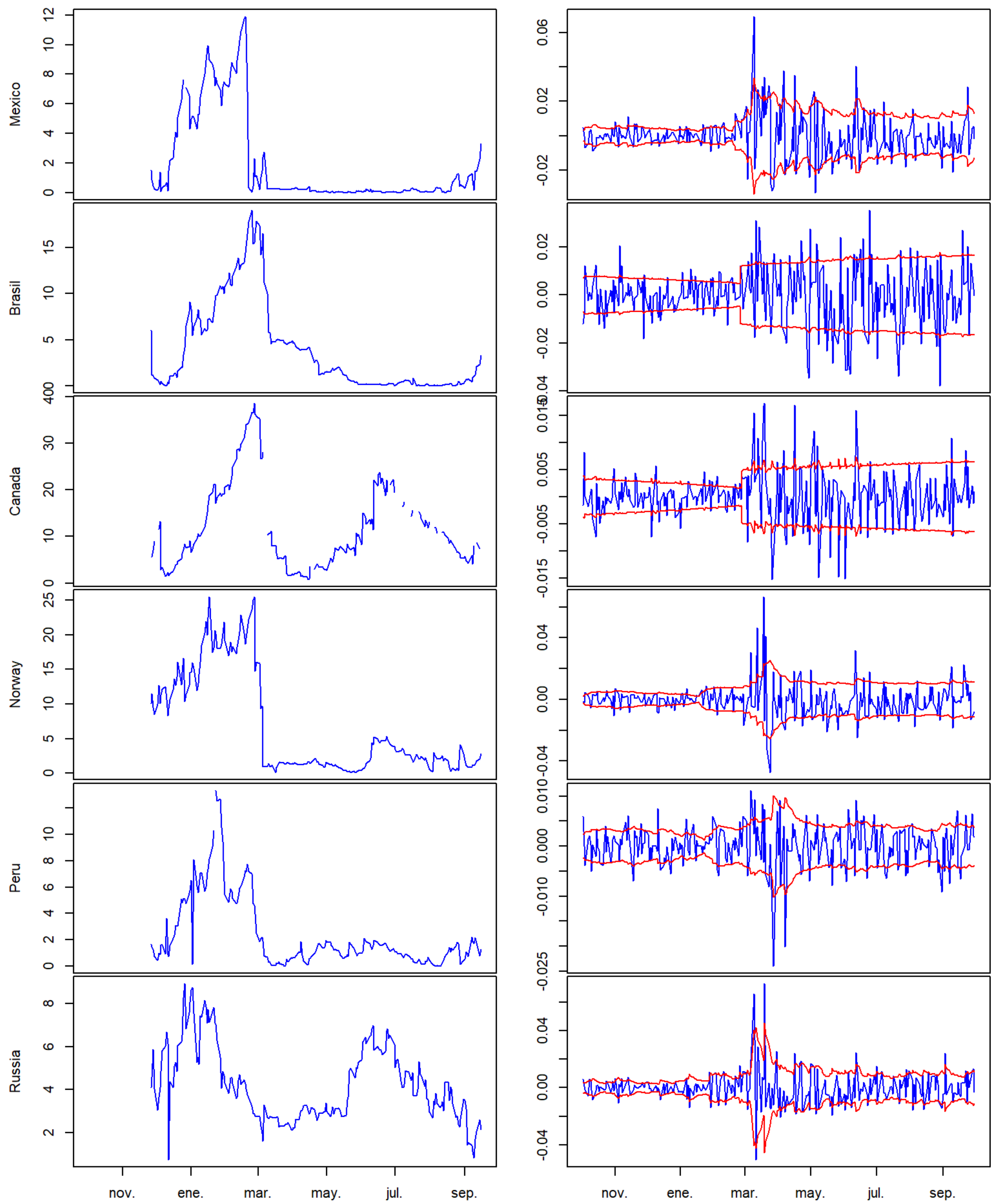

Fig. 2: Left: Test on Currencies to detect the change points.
Right: Roots of the square of the volatilities modelled by trend-GARCH model in red, volatilities in blue.

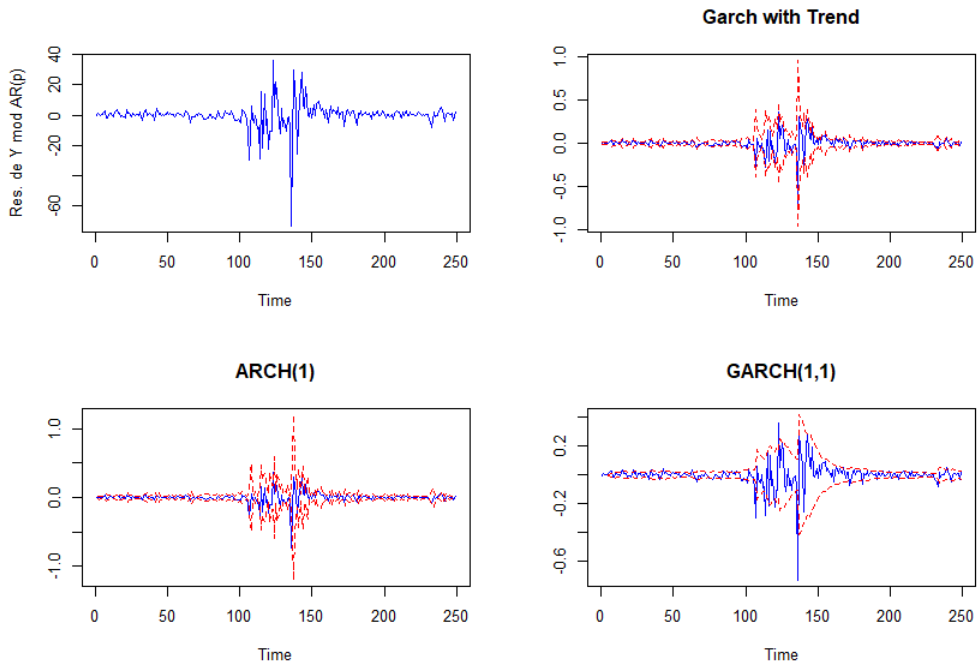

Fig. 3: Res. de Y mod AR(p), Garch with trend,
ARCH(1) and GARCH(1,1) models on WTI

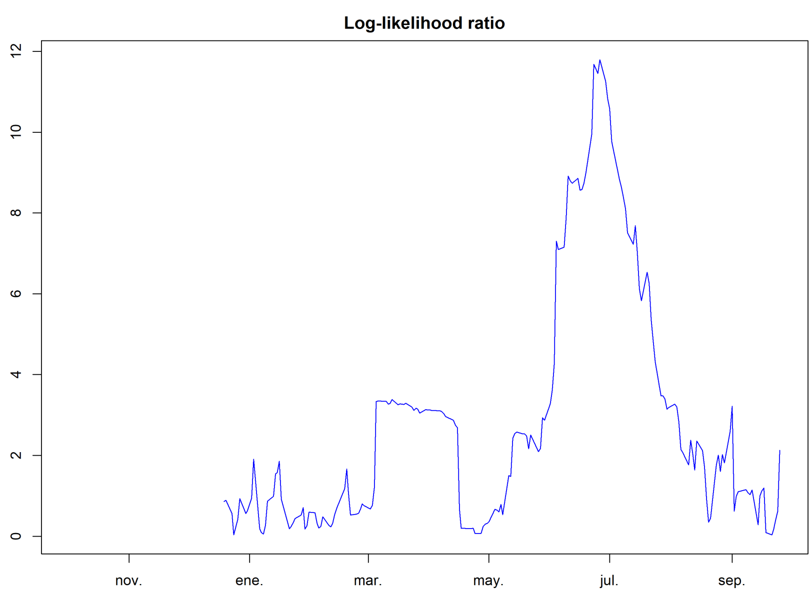

Fig. 4: Log-likelihood ratio statistic for WTI

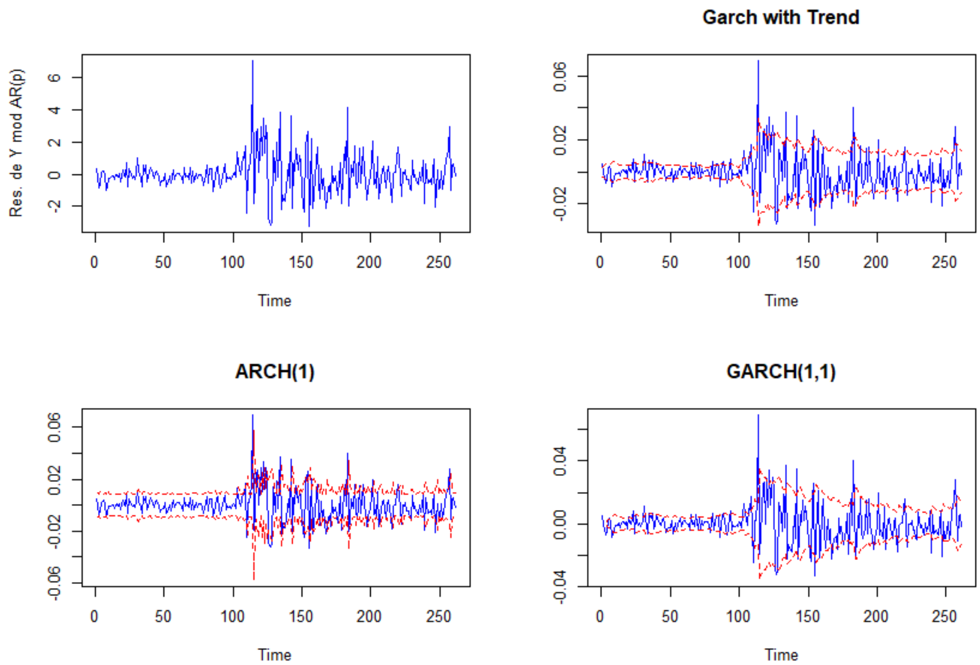

Fig. 5: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on MXN/USD

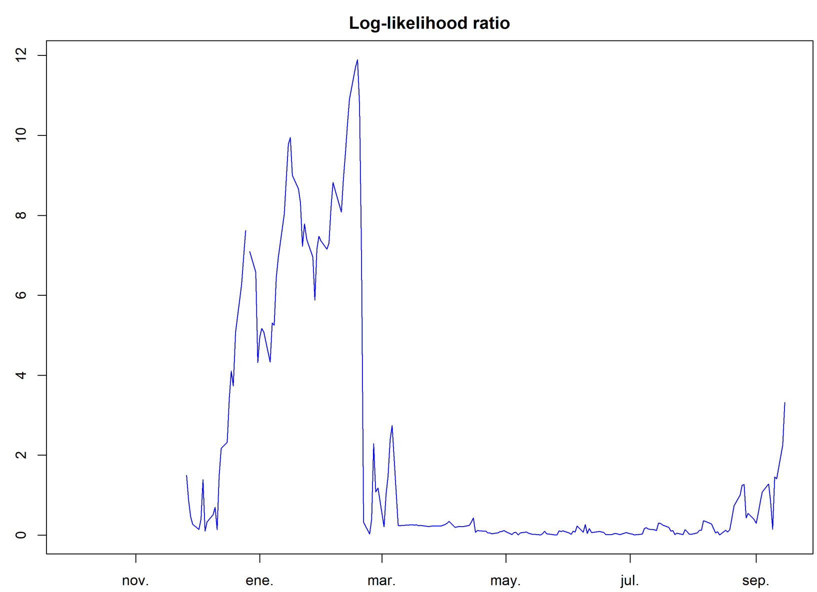

Fig. 6: Log-likelihood ratio statistic for MXN/USD

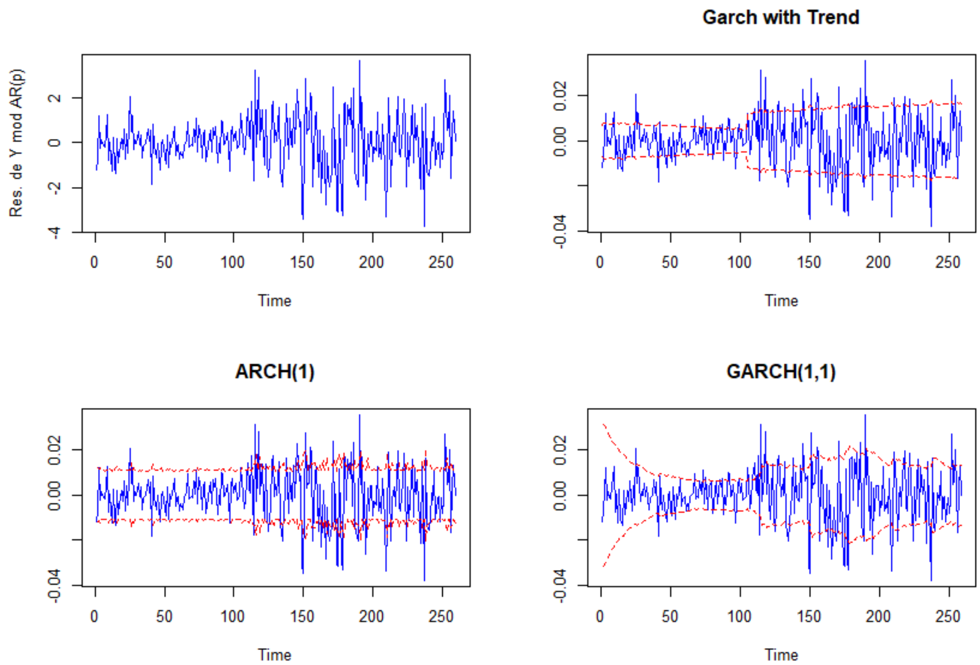

Fig. 7: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on BRL/USD

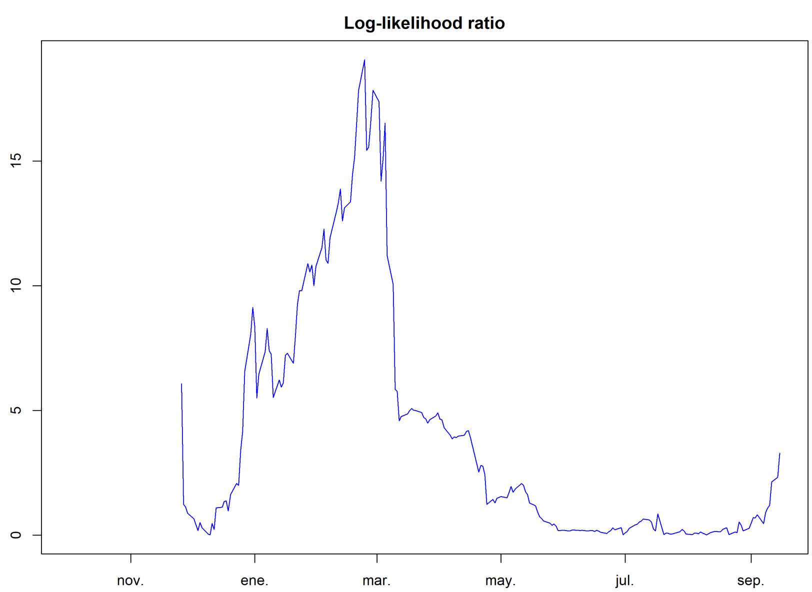

Fig. 8: Log-likelihood ratio statistic for BRL/USD

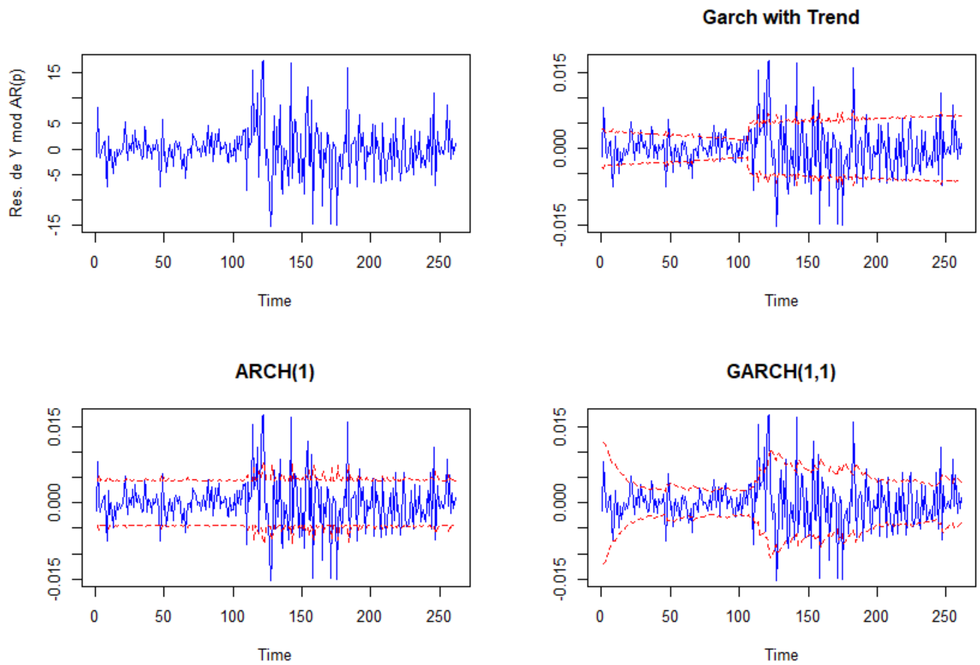

Fig. 9: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on CAD/USD

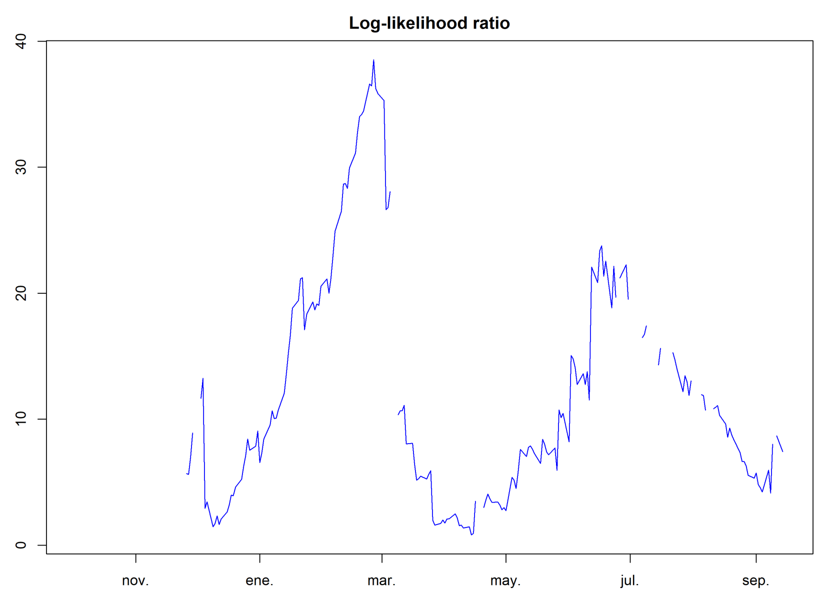

Fig. 10: Log-likelihood ratio statistic for CAD/USD

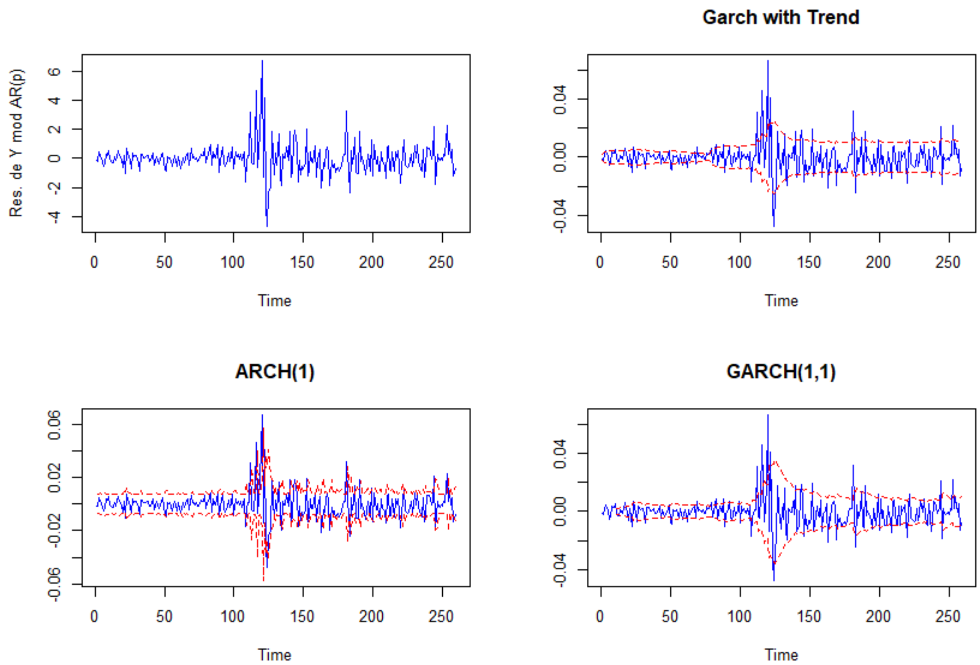

Fig. 11: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on NOK/USD

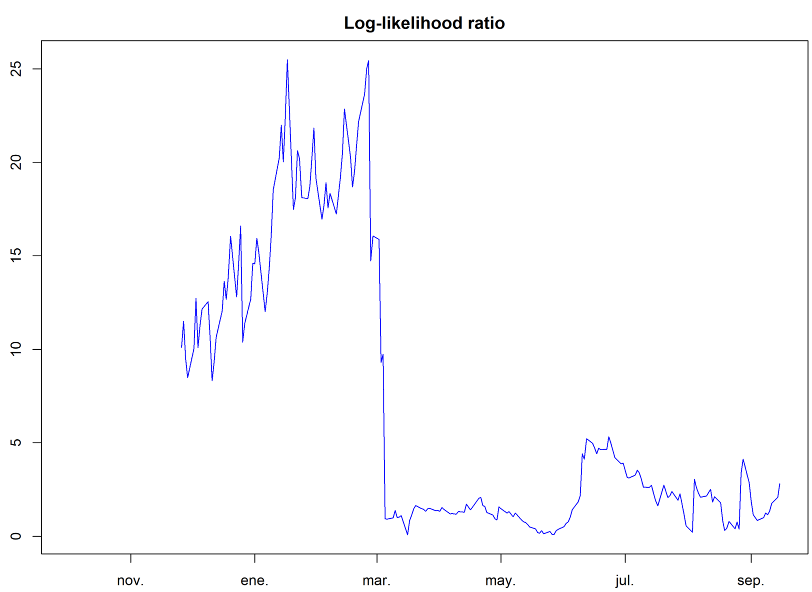

Fig. 12: Log-likelihood ratio statistic for NOK/USD

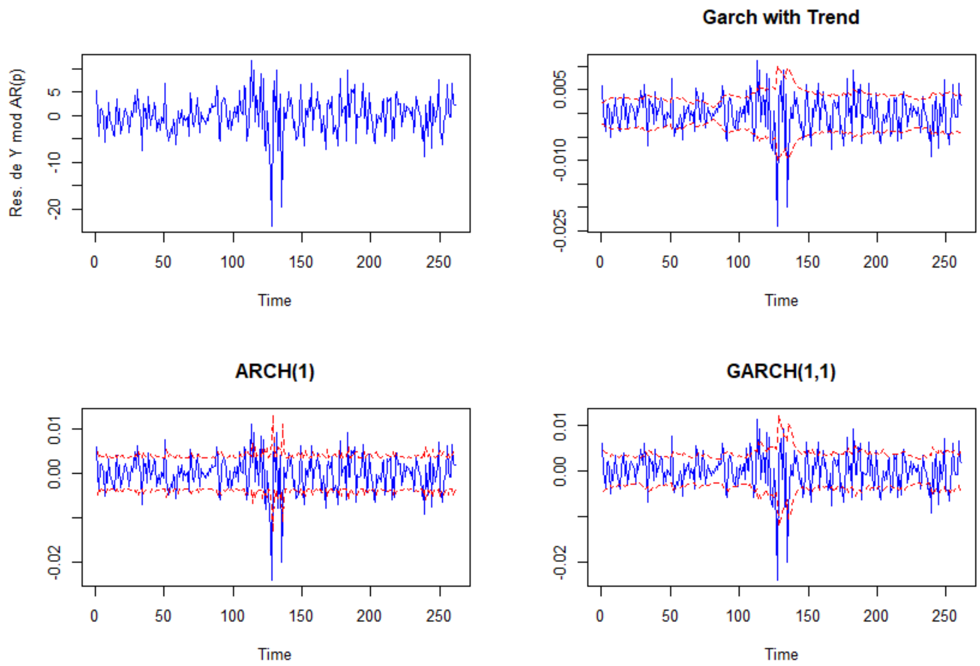

Fig. 13: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on PEN/USD

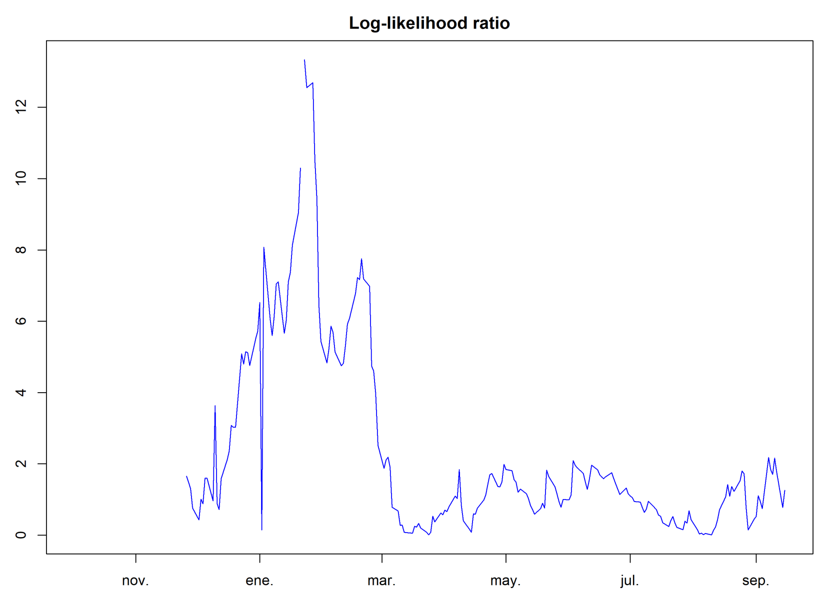

Fig. 14: Log-likelihood ratio statistic for PEN/USD

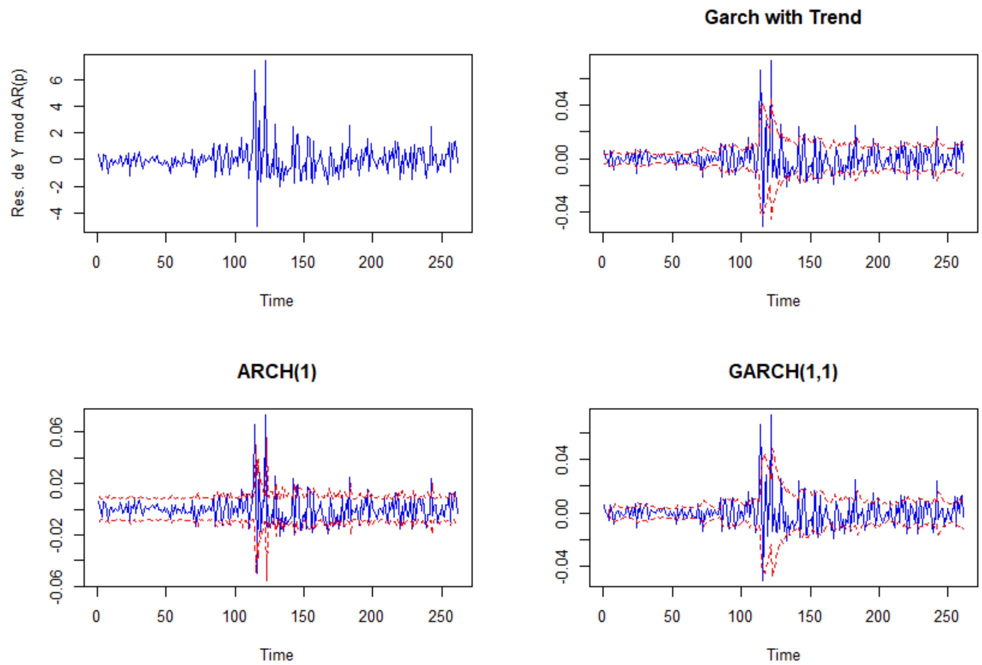

Fig. 15: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on RUB/USD

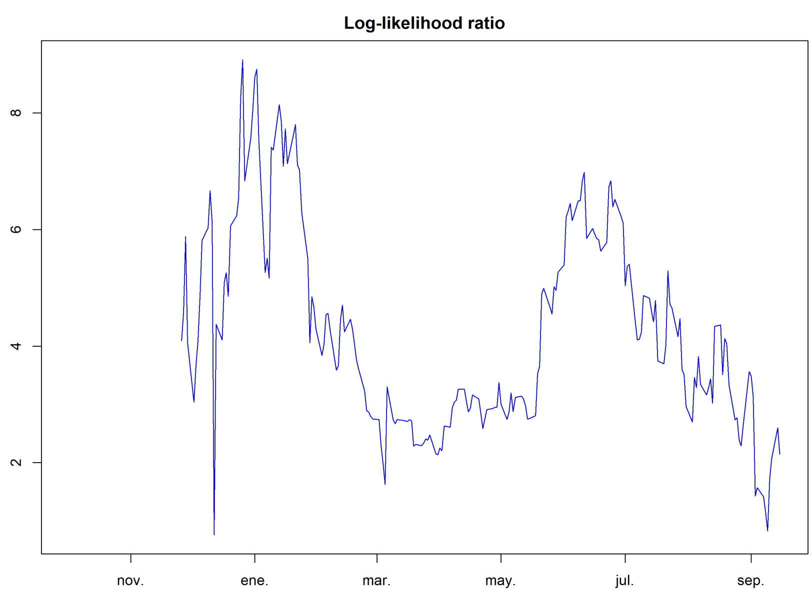

Fig. 16: Log-likelihood ratio statistic for RUB/USD

Appendix B
Market Indexes Tests

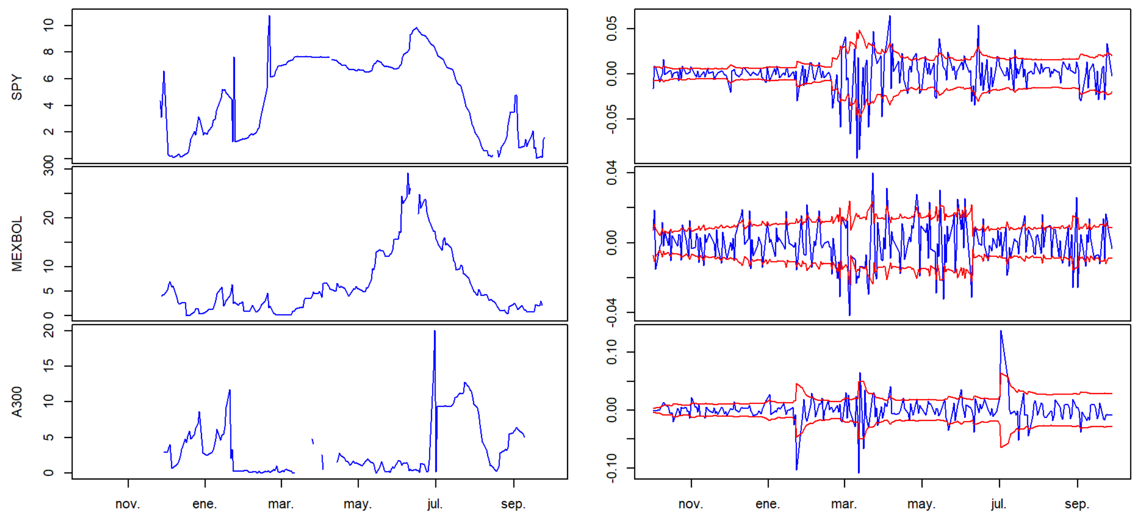

Fig. 17: Left: Test on Market Indexes to detect the change points.
Right: Roots of the square of the volatilities modelled by
trend-GARCH model in red, volatilities in blue.

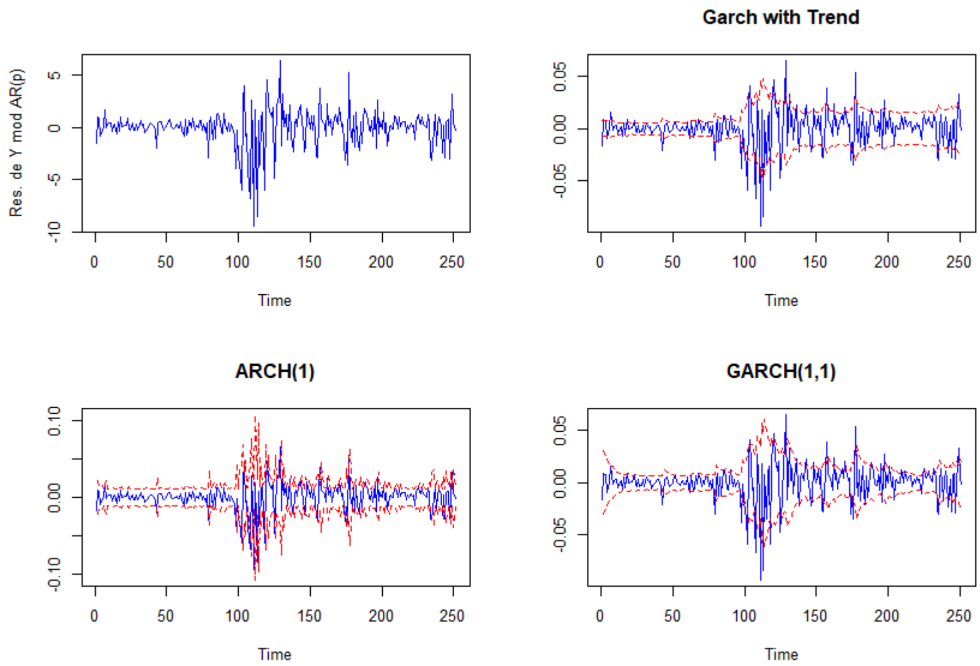

Fig. 18: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on SPY index

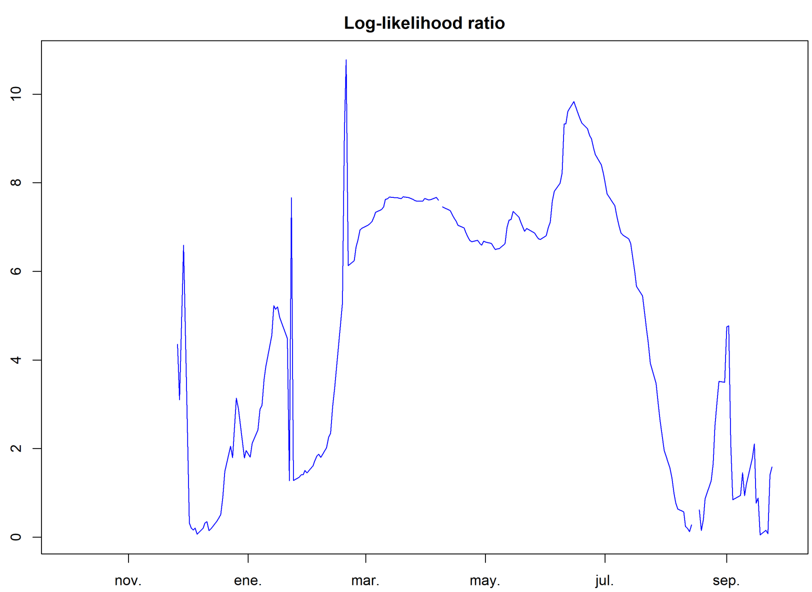

Fig. 19: Log-likelihood ratio statistic for SPY index

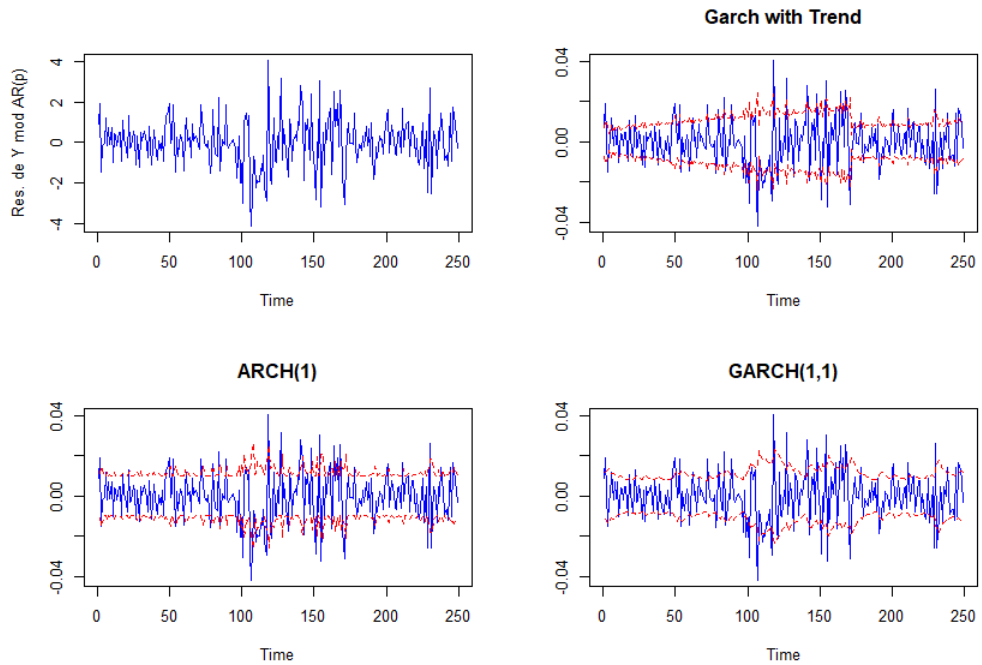

Fig. 20: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on MXX index

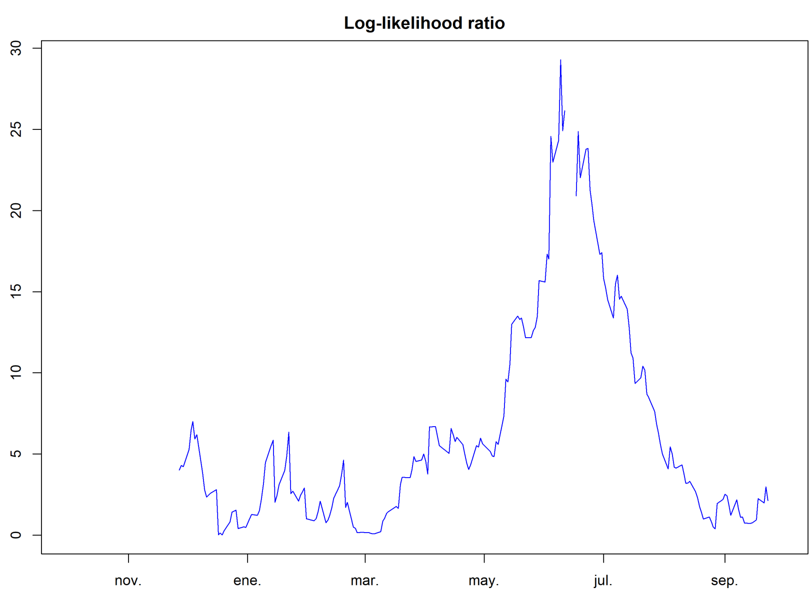

Fig. 21: Log-likelihood ratio statistic for MXX index

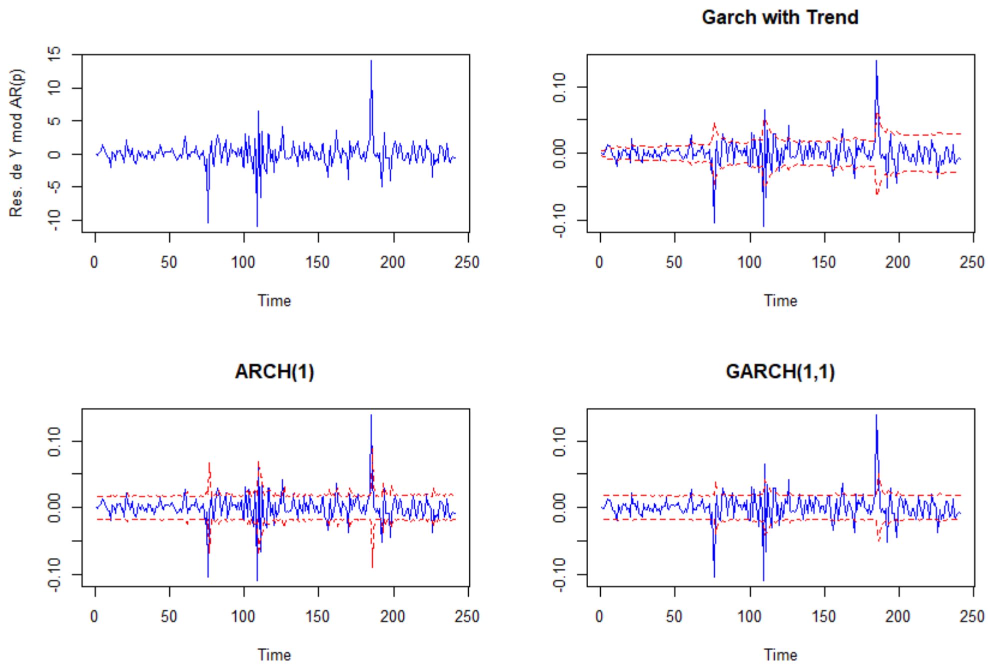

Fig. 22: Res. de Y mod AR(p), Garch with trend, ARCH(1) and GARCH(1,1) models on A300 index

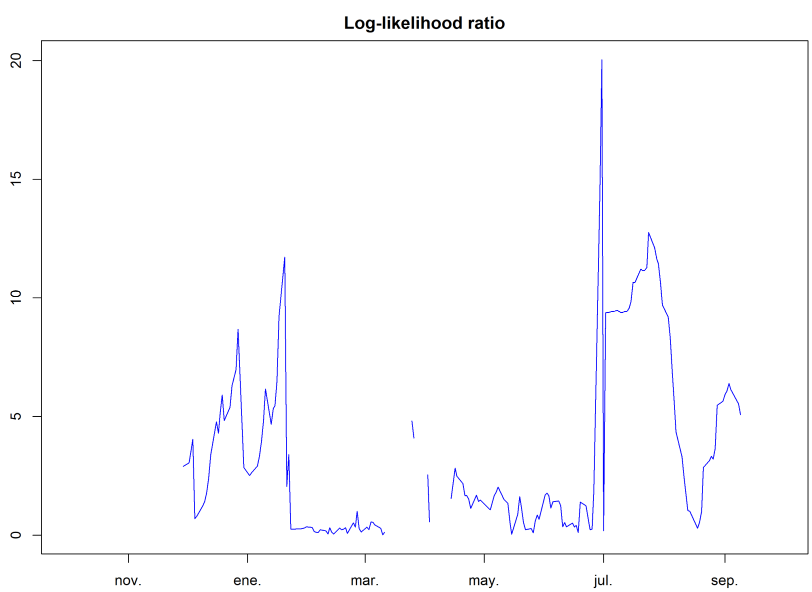

Fig. 23: Log-likelihood ratio statistic for A300 index